JOURNAL OF ALGEBRA 31, 67–72 (1974)

APPENDIX: Degenerate Bilinear Forms

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Communicated by J. Tits

Received April 10, 1973

1.

All vector spaces to be considered are defined over some fixed field F and are supposed to be finite dimensional. A pair (V, Φ) formed by a vector space V and a bilinear form Φ will be called a *bilinear space*; it will be called *nondegenerate* if Φ is so. We denote by Φ_g and Φ_d the maps $x \mapsto \Phi(x, ?)$ and $x \mapsto \Phi(?, x)$ from V to the dual space V^*. Identifying V with its bidual V^{**} we get clearly $\Phi_d = \Phi_g{}^* =$ transpose of Φ_g. Attaching to each bilinear space (V, Φ) the quadruple (V, V^*, Φ_g, Φ_d), we thus get a relation between bilinear spaces and what we shall call *Kronecker modules*: these are by definition quadruples (S, B, g, d) formed by two vector spaces S, B and two linear maps $g, d : S \to B$.

Reversely, we may attach to any Kronecker module (S, B, g, d) the bilinear space $(S \oplus B^*, \Psi)$ with $\Psi((x, \varphi), (y, \psi)) = \psi(gx) + \varphi(dy)$. A bilinear space of this type will be called *neutral*. A more precise study of our two "functors" $(V, \Phi) \mapsto (V, V^*, \Phi_g, \Phi_d)$ and $(S, B, g, d) \mapsto (S \oplus B^*, \Psi)$ will easily furnish the following.

THEOREM. *Any bilinear space is the orthogonal direct sum of a neutral bilinear space and a nondegenerate bilinear space.*

This reduces the study of bilinear forms to the nondegenerate case treated in [3, 4].

2.

Of course, the *orthogonal direct sum* of two bilinear spaces (V_1, Φ_1) and (V_2, Φ_2) is the bilinear space $(V_1 \oplus V_2, \Phi)$ with $\Phi((x_1, x_2), (y_1, y_2)) = \Phi_1(x_1, y_1) + \Phi_2(x_2, y_2)$. Clearly, (V, Φ) is an orthogonal direct sum of two subspaces $(V_1, \Phi \mid V_1)$ and $(V_2, \Phi \mid V_2)$ iff $V = V_1 \oplus V_2$ and $\Phi(V_1, V_2) =$

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$0 = \Phi(V_2, V_1)$. This last condition also means that $\Phi_g(V_1) \subset V_2^{\perp}$ and $\Phi_d(V_1) \subset V_2^{\perp}$. The map $(V_1, V_2) \mapsto (V_1, V_2^{\perp})$ thus gives rise to a bijection between the orthogonal direct sum decompositions of (V, Φ) and the Kronecker submodules (S_1, B_1) of (V, V^*, Φ_g, Φ_d) having the supplementary property that the canonical duality between V and V^* induces a duality between S_1 and B_1; this means in other words that, if $i : S_1 \to V$ and $j : B_1 \to V^*$ denote the inclusions, the composition $j^* \circ i : S_1 \to B_1^*$ has to be bijective. Of course, a *Kronecker submodule* of some (S, B, g, d) is by definition a pair (S_1, B_1) formed by subspaces $S_1 \subset S$ and $B_1 \subset B$ such that $g(S_1) \subset B_1$ and $d(S_1) \subset B_1$.

3.

As seen previously, we have to know more about Kronecker modules. We sum up here some well known facts. Let a morphism

$$(S_1, B_1, g_1, d_1) \to (S_2, B_2, g_2, d_2)$$

between two Kronecker modules be a pair (s, b) of two linear maps $s : S_1 \to S_2$ and $b : B_1 \to B_2$ such that $g_2 s = b g_1$ and $d_2 s = b d_1$. By defining the composition of morphisms in the obvious way, we get an abelian category, where the direct sum of two Kronecker modules is given by

$$(S_1, B_1, g_1, d_1) \oplus (S_2, B_2, g_2, d_2) = (S_1 \oplus S_2, B_1 \oplus B_2, g_1 \oplus g_2, d_1 \oplus d_2).$$

Any Kronecker module may be decomposed into a direct sum of *indecomposable* Kronecker submodules and two such decompositions are related together by the celebrated Krull-Remak-Schmidt theorem, which we state here in its "exchange theorem" form: *Let $M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} M_j'$ be two decompositions of the* (Kronecker) *module M into direct sums of indecomposable submodules. For any subset $I' \subset I$ there is a bijection $\sigma : I \xrightarrow{\sim} J$ such that M_i is isomorphic to $M'_{\sigma(i)}$ for every $i \in I$, and that*

$$M = \Big(\bigoplus_{i \in I'} M_i\Big) \oplus \Big(\bigoplus_{i \notin I'} M'_{\sigma(i)}\Big).$$

This exchange theorem has the following corollary, which will be needed in the sequel: *let Σ be any set of indecomposable Kronecker modules*; *call a Kronecker module M of type Σ if any indecomposable direct summand of M is isomorphic to some module of Σ*; *call M of type Σ' if no direct summand is of type Σ. Then for any two decompositions $M = M_1 \oplus M_1' = M_2 \oplus M_2'$ with M_1, M_2 of type Σ and M_1', M_2' of type Σ', we also get direct decompositions $M = M_1 \oplus M_2' = M_2 \oplus M_1'$.*

The indecomposable Kronecker modules have been classified by Kronecker (this is the so called problem of simultaneous reduction of two matrices [2]; see also [1]). They are of the following types:

I. $M(\sigma) = (V, V, \mathrm{id}_V, \sigma)$, where σ is an automorphism of V under which no two nonzero complements of V are stable.

II. $M_n{}^0 = (F^n, F^n, \mathrm{id}_{F^n}, \sigma)$, $n \geqslant 1$, where σ is defined by the Jordan matrix

$$\begin{bmatrix} 0 & 0 & \cdots & & 0 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

*II** $M_n{}^\infty = (F^n, F^n, \sigma, \mathrm{id}_{F^n})$, $n \geqslant 1$, σ as in II.

III. $\hat{n} = (F^n, F^{n+1}, g, d)$, $n \geqslant 0$, with $g(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0)$ and $d(x_1, \ldots, x_n) = (0, x_1, \ldots, x_n)$.

*III**. $\underset{\vee}{n} = (F^{n+1}, F^n, g, d)$, $n \geqslant 0$, with $g(y_0, \ldots, y_n) = (y_1, \ldots, y_n)$ and $d(y_0, \ldots, y_n) = (y_0, \ldots, y_{n-1})$.

4.

Not every Kronecker module can be constructed by means of some bilinear space. Indeed, if $M = (S, B, g, d)$ is a Kronecker module, set $M^* = (B^*, S^*, d^*, g^*) =$ the *dual* of M. For any bilinear space (V, Φ) the associated Kronecker module $M(V, \Phi) = (V, V^*, \Phi_g, \Phi_d)$ is clearly *selfdual*, i.e., isomorphic to its dual. In fact we identify $M(V, \Phi)$ with $M(V, \Phi)^*$ by means of the obvious canonical isomorphism.

Now consider a selfdual set Σ of indecomposable Kronecker modules (that is, for any $M \in \Sigma$, Σ also contains a module isomorphic to M^*). Let $M(V, \Phi) = M \oplus M'$ be a direct decomposition of $M(V, \Phi)$ into a summand M of type Σ and a summand M' of type Σ'. Then $M(V, \Phi) = M(V, \Phi)^* = M^* \oplus M'^*$ is another, possibly different, decomposition of $M(V, \Phi)$ into a Σ-part and a Σ'-part. As seen in Section 3, $M(V, \Phi) = M \oplus M'^*$ is again a direct decomposition, a fact which admits the following equivalent interpretation: let $i : S \to V$ and $j : B \to V^*$ be the inclusions, where $M = (S, B, g, d)$. Then (j^*, i^*): $M(V, \Phi) \to M^*$ is the projection along M'^*. This projection induces an isomorphism (j^*i, i^*j): $M \xrightarrow{\sim} M^*$, so that $j^*i : S \to B^*$ must be bijective and $V = S \oplus B^\perp$ an orthogonal direct sum decomposition of V relatively to the bilinear form Φ (see Section 2). This reduces the study of bilinear spaces to the cases where $M(V, \Phi)$ is either of type Σ or of type Σ'.

Taking in a first step for Σ the set of all Kronecker modules $M(\sigma)$ of type I, Section 3, we can separate the nondegenerate case treated in [3] from the "totally degenerate" case, which by a new choice of Σ can now be reduced to the cases where $M(V, \Phi)$ is of type $\{M_n{}^0, M_n{}^\infty\}$ with $n \geqslant 1$ or of type $\{\hat{n}, \underset{\vee}{n}\}$ with $n \geqslant 0$. We study these cases in Sections 5 and 6.

5.

Now we consider a bilinear space (V, Φ) *with associated Kronecker module* $M(V, \Phi) = M(V, \Phi)^* = (V, V^*, \Phi_g, \Phi_d)$ *of type* $\{M_n{}^0, M_n{}^\infty\}$. As $M(V, \Phi)$ is selfdual, it must be isomorphic to some $(M_n{}^0 \oplus M_n{}^\infty)^N$, $N \geqslant 0$. Set $M_n^{0N} = (S, B, g, d) = M$ and take any isomorphism

$$(s, b) : M \oplus M^* \xrightarrow{\sim} M(V, \Phi)$$

(notice of course that $M_n^{0*} \xrightarrow{\sim} M_n{}^\infty$). Its transpose gives rise to an isomorphism $(b^*, s^*) : M(V, \Phi) = M(V, \Phi)^* \to (M \oplus M^*)^* \xrightarrow{\sim} M \oplus M^*$, if we identify $B \oplus S^*$ with $(S \oplus B^*)^*$:

$$\begin{tikzcd}
S \oplus B^* \arrow[r, "s", "\sim"'] \arrow[d, shift right, "g \oplus d^*"'] \arrow[d, shift left, "d \oplus g^*"] & V \arrow[r, "b^*", "\sim"'] \arrow[d, shift right, "\Phi_g"'] \arrow[d, shift left, "\Phi_d"] & S \oplus B^* \arrow[d, shift right, "g \oplus d^*"'] \arrow[d, shift left, "d \oplus g^*"] \\
B \oplus S^* \arrow[r, "b", "\sim"'] & V^* \arrow[r, "s^*", "\sim"'] & B \oplus S^*.
\end{tikzcd}$$

The composition $(b^*s,\ s^*b) : M \oplus M^* \xrightarrow{\sim} M \oplus M^*$ must be of the form $\mu \oplus \nu$, where $\mu = (\sigma, \beta)$: $M \xrightarrow{\sim} M$ and $\nu : M^* \xrightarrow{\sim} M^*$ are automorphisms. This is so because any morphism $M_n{}^0 \to M_n{}^\infty$ or $M_n{}^\infty \to M_n{}^0$ is zero, the same being therefore true for any morphism $M \to M^*$ or $M^* \to M$. Moreover, as $\nu^* \oplus \mu^* = (\mu \oplus \nu)^* = (b^*s, s^*b)^* = (b^*s, s^*b) = \mu \oplus \nu$, we get $\nu = \mu^*$.

Now modify the initial choice of (s, b) replacing s by $s\left[\begin{smallmatrix}\sigma^{-1} & 0 \\ 0 & \mathrm{id}\end{smallmatrix}\right]$ and b by $b\left[\begin{smallmatrix}\beta^{-1} & 0 \\ 0 & \mathrm{id}\end{smallmatrix}\right]$. Then b^* and s^* are replaced respectively by $\left[\begin{smallmatrix}\mathrm{id} & 0 \\ 0 & \beta^{*-1}\end{smallmatrix}\right]b$ and $\left[\begin{smallmatrix}\mathrm{id} & 0 \\ 0 & \sigma^{*-1}\end{smallmatrix}\right]s$, whereas μ becomes the identity. Therefore we may assume from the beginning that $\mu = (\mathrm{id}_S , \mathrm{id}_B) = \mathrm{id}_M$, which means that $b = s^{*-1}$, or equivalently that $s : V \xrightarrow{\sim} S \oplus B^*$ induces an isomorphism of (V, Φ) onto the bilinear space $(S \oplus B^*, \Psi)$ such that $\Psi_g = g \oplus d^*$. This is the neutral bilinear space considered in Section 1.

6.

Finally we consider a bilinear space (V, Φ) *with associated Kronecker module* $M(V, \Phi)$ *of type* $\{{}^{\wedge}_{n}, {}^{n}_{\vee}\}$. The case $n = 0$ is uninteresting, because in this case Φ is zero. Suppose therefore that $n \geqslant 1$ and start as in Section 5 : as $M(V, \Phi)$ is selfdual, it is isomorphic to some $({}^{\wedge}_{n} \oplus {}^{n}_{\vee})^N$, $N \geqslant 0$. Set

$$ {}^{\wedge}_{n}{}^{N} = (S, B, g, d) = M $$

and take any isomorphism (s, b): $M \oplus M^* \simeq M(V, \Phi)$ giving rise as in Section 5 to the diagram

$$\begin{tikzcd}
S \oplus B^* \arrow[r, "s", "\sim"'] \arrow[d, shift right, "g \oplus d^*"'] \arrow[d, shift left, "d \oplus g^*"] & V \arrow[r, "b^*", "\sim"'] \arrow[d, shift right, "\Phi_g"'] \arrow[d, shift left, "\Phi_d"] & S \oplus B^* \arrow[d, shift right, "g \oplus d^*"'] \arrow[d, shift left, "d \oplus g^*"] \\
B \oplus S \arrow[r, "b", "\sim"'] & V \arrow[r, "s^*", "\sim"'] & B \oplus S^*.
\end{tikzcd} \qquad \text{(a)}$$

The composition (b^*s, s^*b): $M \oplus M^* \simeq M \oplus M^*$ must now be of the form $\left[\begin{smallmatrix} \mu & 0 \\ \pi & \nu \end{smallmatrix}\right]$, where $\mu = (\sigma, \beta)$: $M \to M$ and $\nu : M^* \to M^*$ are automorphisms and $\pi = (\rho, \tau)$ is a morphism $M \to M^*$. This is so because any morphism ${}^{n}_{\vee} \to {}^{\wedge}_{n}$ or $M^* \to M$ is zero (but now there are nonzero morphisms $M \to M^*$). Moreover, as $\left[\begin{smallmatrix} \nu^* & 0 \\ \pi^* & \mu^* \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mu & 0 \\ \pi & \nu \end{smallmatrix}\right]^* = (b^*s, s^*b)^* = (b^*s, s^*b) = \left[\begin{smallmatrix} \mu & 0 \\ \pi & \nu \end{smallmatrix}\right]$, we get $\nu = \mu^*$, and $\pi = \pi^*$ or equivalently $\rho = \tau^*$. Replacing again s by $s\left[\begin{smallmatrix} \sigma^{-1} & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right]$ and b by $b\left[\begin{smallmatrix} \beta^{-1} & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right]$, we see as in Section 5 that we may assume from the beginning that $\sigma = \mathrm{id}_S$ and $\beta = \mathrm{id}_B$, so that $(b^*s, s^*b) = \left[\begin{smallmatrix} 1 & 0 \\ \pi & \mathrm{id} \end{smallmatrix}\right]$, with $\pi = (\tau^*, \tau)$: $(S, B, g, d) \to (B^*, S^*, d^*, g^*)$. From diagram (a) we deduce an isomorphism of Kronecker modules

$$\begin{tikzcd}
S \oplus B^* \arrow[r, "s", "\sim"'] \arrow[d, shift right, "G"'] \arrow[d, shift left, "D"] & V \arrow[d, shift right, "\Phi_g"'] \arrow[d, shift left, "\Phi_d"] \\
B \oplus S^* \arrow[r, "s^{*-1}", "\sim"'] & V
\end{tikzcd} \qquad \text{(b)}$$

where $G = s^*b(g \oplus d^*) = \left[\begin{smallmatrix} g & 0 \\ \tau g & d^* \end{smallmatrix}\right]$ and $D = s^*b(d \oplus g^*) = \left[\begin{smallmatrix} d & 0 \\ \tau d & g^* \end{smallmatrix}\right]$. This means that our bilinear module (V, Φ) is isomorphic to $(S \oplus B^*, \Psi^A)$ with $\Psi_g{}^A = G$, or equivalently

$$\Psi^A((x, \varphi), (y, \psi)) = \psi(gx) + \varphi(dy) + A(x, y),$$

if we set $A(x, y) = (\tau g(x))(y)$. The theorem of Section 1 still to be proved therefore follows from the

LEMMA. *Suppose that* $M = (S, B, g, d) = {}_{n}^{\wedge N}$, $N \geqslant 0$. *For any bilinear form* A *on* S, *the bilinear space* $(S \oplus B^*, \Psi^A)$ *defined above is isomorphic to* $(S \oplus B^*, \Psi^0)$ *and is therefore neutral.*

In fact, consider any linear automorphism i of $S \oplus B^*$ of the form $\left[\begin{smallmatrix} \mathrm{id} & 0 \\ T & \mathrm{id} \end{smallmatrix}\right]$, where $T: S \to B^*$ is a linear map. For any $x, y \in S$ and $\varphi, \psi \in B^*$ we get

$$\begin{aligned} \Psi^A(i(x, \varphi), i(y, \psi)) &= \Psi^A((x, Ty + \varphi), (y, Ty + \psi)) \\ &= \psi(gx) + (Ty)(gx) + \varphi(dy) + (Tx)(dy) + A(x, y) \\ &= \Psi^{A'}((x, \varphi), (y, \psi)), \end{aligned}$$

where $A'(x, y) = A(x, y) + (T^*gx)(y) + (d^*Tx)(y)$.

It remains to show that $A' = 0$ for a suitable choice of T, or equivalently that any linear map $L: S \to S^*$ has the form $L = T^*g + d^*T$ for some $T: S \to B^*$. To prove that, we give a conceptual translation, due to J. Tits, of a previous computation: consider the retraction $d' = \delta^N: B \to S$ of d, where $\delta: F^{n+1} \to F^n$ maps $(y_0, y_1, \dots, y_n)$ onto $(y_1, \dots, y_n)$. We clearly get $(d'g)^n = 0$. Look for a T of the form $T = d'^*t$, with $t: S \to S^*$. Equation $L = T^*g + d^*T$ is then replaced by $L = t^*d'g + t = (1 + \nu)(t)$, with $\nu(t) = t^*d'g$. Since $\nu^2(t) = (d'g)^*t(d'g)$ and $\nu^{2n}(t) = (d'g)^{*n}t(d'g)^n = 0$,the last equation has the unique solution

$$t = (1 + \nu)^{-1}(L) = (1 - \nu + \nu^2 - \cdots)(L) = L - \nu(L) + \nu^2(L) - \nu^3(L) \cdots$$

7.

Our method is quite similar to the way followed originally by Kronecker, who attached to a bilinear space (V, Φ) the Kronecker module

$$(V, V^*, A_g, S_g),$$

where A and S are respectively the antisymmetric and the symmetric part of Φ. But of course, A and S are only defined if $\operatorname{char} F \neq 2$, whereas $(V, V^*, \Phi_g, \Phi_d) = M(V, \Phi)$ is defined for any characteristic.

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